A Spurious Minimum for the Logarithmic Fekete Problem on S^2 using Gröbner Bases and Interval Arithmetic via *msolve*

Matías Valdés

Albatross - Paris

September 3, 2026



SEDE TACUAREMBÓ

CENUR NORESTE

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

Outline

Introduction - Fekete Problem

Spurious Minima in S^2 - Numerical Candidates

Coordinate Ring of Candidate for $n = 16$

Hessian at Candidate - Numerical Eigenvalues

Representation of Group of Symmetries of Candidate

Block Diagonalizing the Hessian

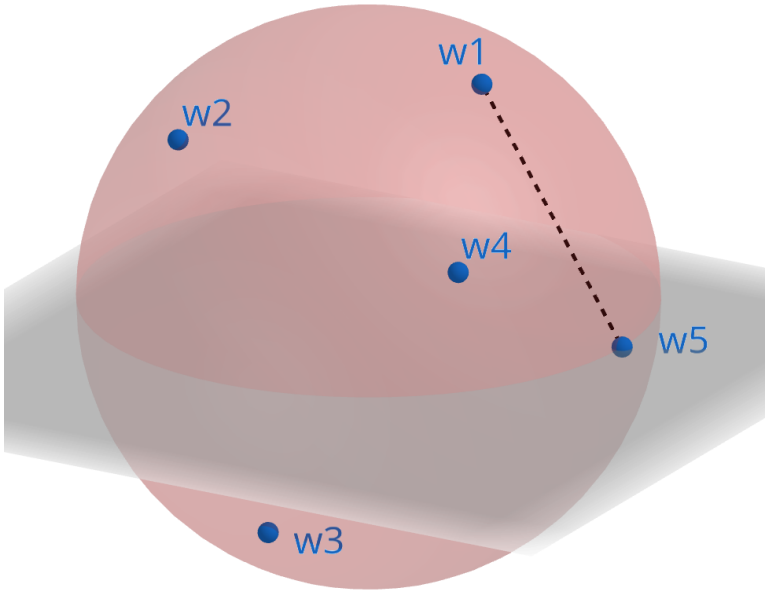
Classifying the Diagonal Blocks

Fekete Problem

Position n points on the sphere $S^d \subseteq \mathbb{R}^{d+1}$, so as to maximize the product of the distances between pairs of points:

$$\max \prod_{1 \le i < j \le n} \|w_i - w_j\|,$$

$$w_i \in S^d \subseteq \mathbb{R}^{d+1}.$$

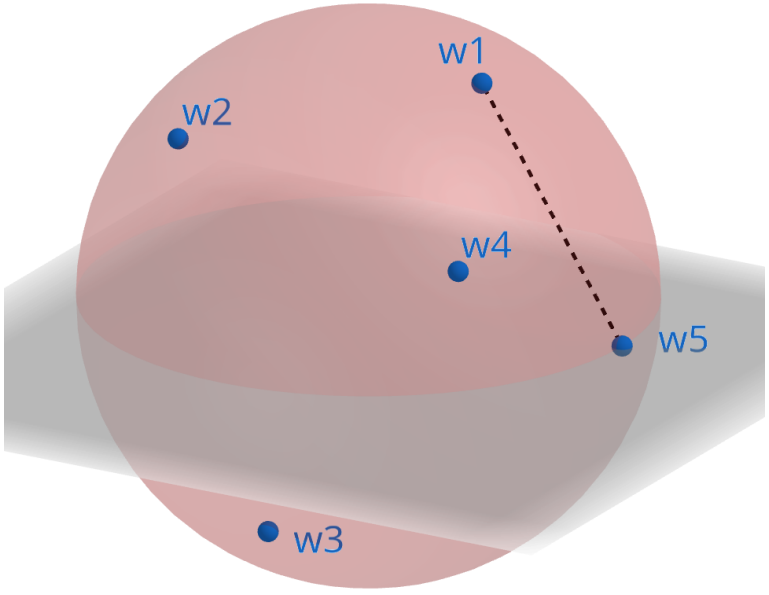


Fekete Problem

Position n points on the sphere $S^d \subseteq \mathbb{R}^{d+1}$, so as to maximize the product of the distances between pairs of points:

$$\max \prod_{1 \le i < j \le n} \|w_i - w_j\|,$$

$$w_i \in S^d \subseteq \mathbb{R}^{d+1}.$$



Equivalently: place the points to minimize the Logarithmic Energy

$$E_{\log}(w) := -\sum_{1 \le i < j \le n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subseteq \mathbb{R}^{d+1}.$$

Why caring about this problem?

Fekete problem: minimize the logarithmic energy on S^d:

$$E_{\log}(w) := -\sum_{1 \le i < j \le n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subseteq \mathbb{R}^{d+1}.$$

Why caring about this problem?

Fekete problem: minimize the logarithmic energy on S^d:

$$E_{\log}(w) := -\sum_{1\le i<j\le n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subseteq \mathbb{R}^{d+1}.$$

Bezout III - Shub and Smale (1993) [6]

Fekete solutions (projected to the plane) are useful for initializing homotopy algorithms. Even solutions up to a logarithmic factor.

Why caring about this problem?

Fekete problem: minimize the logarithmic energy on S^d:

$$E_{\log}(w) := -\sum_{1 \le i < j \le n} \log\left(\|w_i - w_j\|^2\right), \quad w_i \in S^d \subseteq \mathbb{R}^{d+1}.$$

Bezout III - Shub and Smale (1993) [6]

Fekete solutions (projected to the plane) are useful for initializing homotopy algorithms. Even solutions up to a logarithmic factor.

Smale's 7th problem for the next Century (1998) [7]

Give an "efficient" algorithm that, for each number of points n, finds $\hat{w} \in (S^2)^n$ with optimal value up to a logarithmic factor:

$$E_{\log}(\hat{w}) - E^*_{\log} \le c \log(n), \quad \text{for a universal constant } c.$$

What is known about the Fekete problem

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.
- **Optimal value** known asymptotically, up to linear term [1]:

$$\text{in } S^2\text{:} \quad E^*_{\log} = \left(\frac{1}{2} - \log 2\right)\frac{n^2}{2} - \frac{n \log n}{4} + Cn + o(n), \quad C = ?$$

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.
- **Optimal value** known asymptotically, up to linear term [1]:

$$\text{in } S^2\text{:} \quad E^*_{\log} = \left(\frac{1}{2} - \log 2\right)\frac{n^2}{2} - \frac{n \log n}{4} + Cn + o(n), \quad C = ?$$

- Any critical configuration satisfies: $\sum_{i=1}^{n} w_i = \vec{0}$.

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.
- **Optimal value** known asymptotically, up to linear term [1]:

$$\text{in } S^2: \quad E^*_{\log} = \left(\frac{1}{2} - \log 2\right)\frac{n^2}{2} - \frac{n \log n}{4} + Cn + o(n), \quad C = ?$$

- Any critical configuration satisfies: $\sum_{i=1}^{n} w_i = \vec{0}$.
- Has **saddle-points** [3, 4], and degenerate global minima [2].

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.
- **Optimal value** known asymptotically, up to linear term [1]:

$$\text{in } S^2\text{:} \quad E^*_{\log} = \left(\frac{1}{2} - \log 2\right)\frac{n^2}{2} - \frac{n \log n}{4} + Cn + o(n), \;\; C = ?$$

- Any critical configuration satisfies: $\sum_{i=1}^{n} w_i = \vec{0}$.
- Has **saddle-points** [3, 4], and degenerate global minima [2].
- Cannot have local **maxima**.

What is known about the Fekete problem

- **Optimal configurations** known for few values of n and d.
- In S^2, optimals are known only for: $n = 2, 3, 4, 5, 6$ and 12.
- **Optimal value** known asymptotically, up to linear term [1]:

$$\text{in } S^2\text{:} \quad E^*_{\log} = \left(\frac{1}{2} - \log 2\right)\frac{n^2}{2} - \frac{n \log n}{4} + Cn + o(n), \quad C = ?$$

- Any critical configuration satisfies: $\sum_{i=1}^{n} w_i = \vec{0}$.
- Has **saddle-points** [3, 4], and degenerate global minima [2].
- Cannot have local **maxima**.
- What about **spurious minima**? (i.e. non global local minima)

Logarithmic Fekete problem - Spurious Minima

- Only one spurious minimum known [3, 4]: for $n = 6$ in S^3

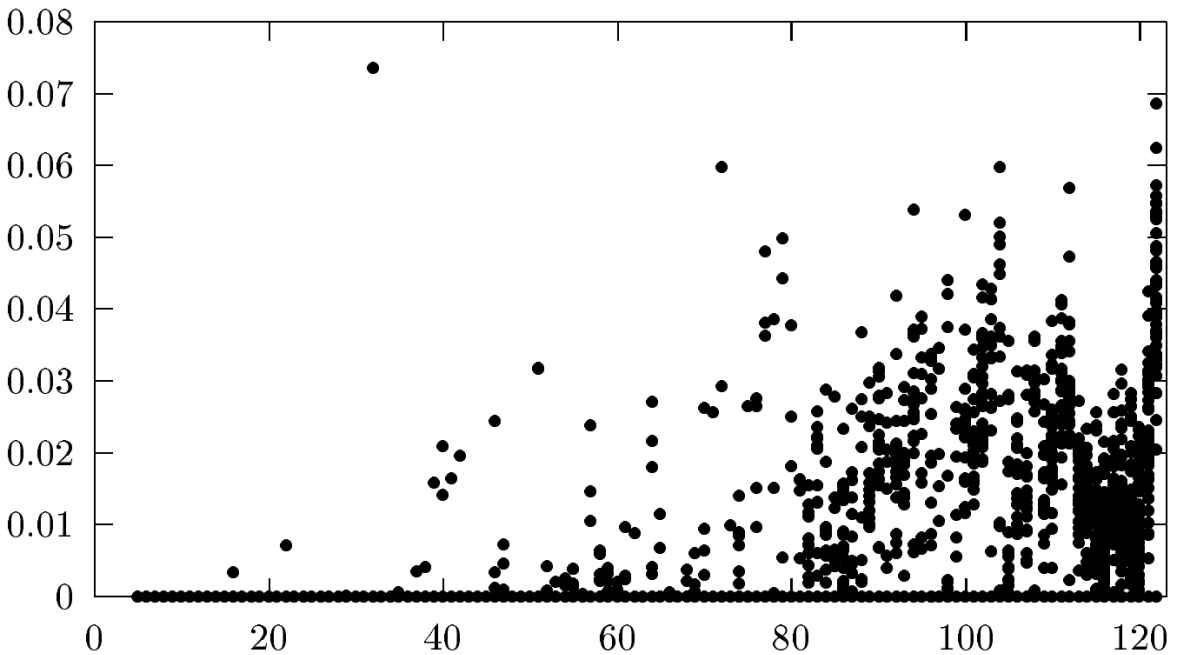
Logarithmic Fekete problem - Spurious Minima

- Only one spurious minimum known [3, 4]: for $n = 6$ in S^3
- Numerically: number of spurious minima in S^2 increases "dramatically" with n. Rakhmanov, Saff, Zhou (1995) [5]

Logarithmic Fekete problem - Spurious Minima

- Only one spurious minimum known [3, 4]: for $n = 6$ in S^3
- Numerically: number of spurious minima in S^2 increases "dramatically" with n. Rakhmanov, Saff, Zhou (1995) [5]

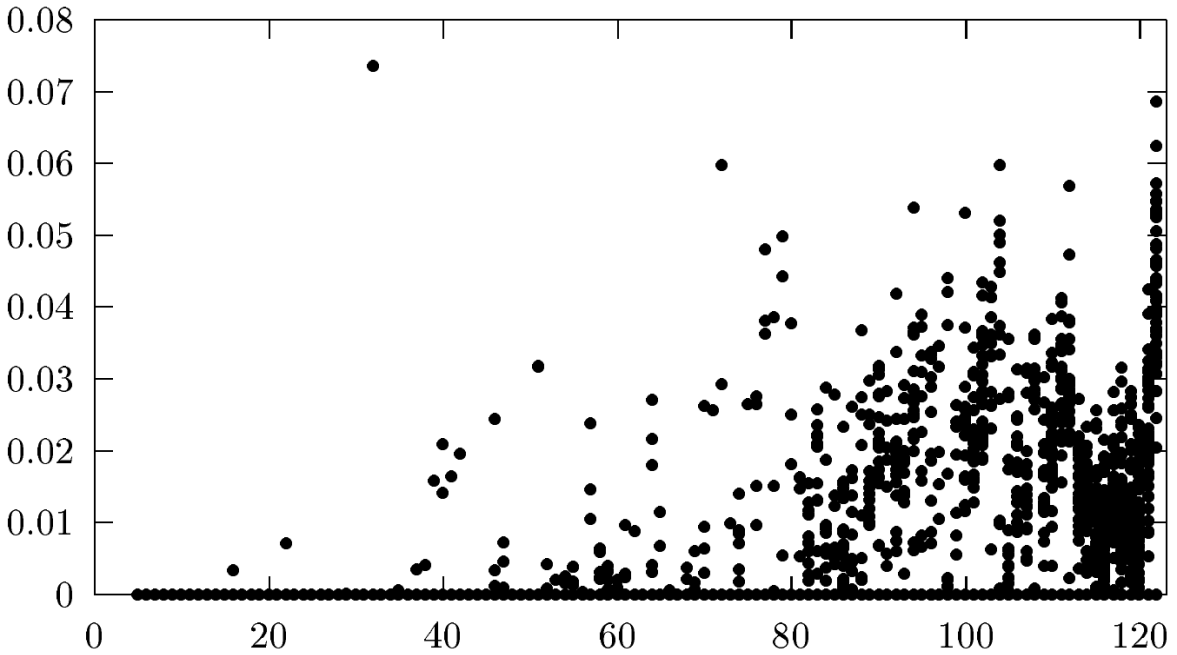
Figure: Energy difference: local - global (numerical) minimum.



Logarithmic Fekete problem - Spurious Minima

- Only one spurious minimum known [3, 4]: for $n = 6$ in S^3
- Numerically: number of spurious minima in S^2 increases "dramatically" with n. Rakhmanov, Saff, Zhou (1995) [5]

Figure: Energy difference: local - global (numerical) minimum.

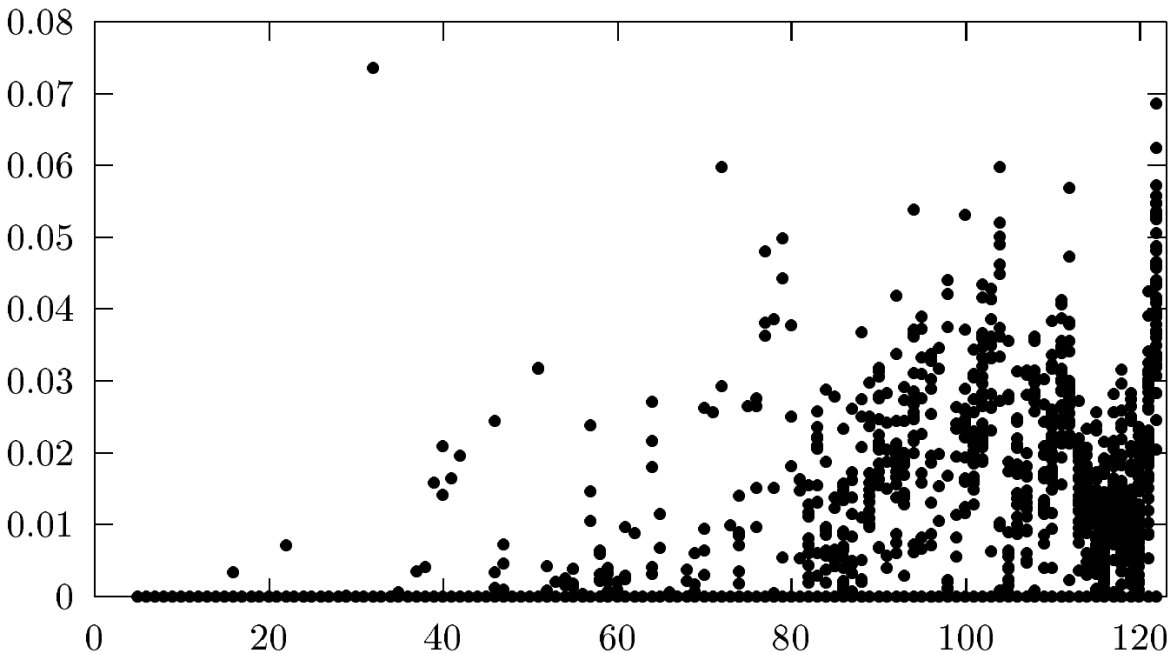


- Rigorously: no spurious minimum on S^2 is known

Logarithmic Fekete problem - Spurious Minima

- Only one spurious minimum known [3, 4]: for $n = 6$ in S^3
- Numerically: number of spurious minima in S^2 increases "dramatically" with n. Rakhmanov, Saff, Zhou (1995) [5]

Figure: Energy difference: local - global (numerical) minimum.



- Rigorously: no spurious minimum on S^2 is known
- We give rigorous proof of spurious minimum on S^2, for $n = 16$

Spurious Minima in S^2 - Numerical Candidates

- Apply Gradient Descent to search for candidates.

Spurious Minima in S^2 - Numerical Candidates

- Apply Gradient Descent to search for candidates.
- $n = 16$ points: smallest with evidence of spurious minima.

Spurious Minima in S^2 - Numerical Candidates

- Apply Gradient Descent to search for candidates.
- $n = 16$ points: smallest with evidence of spurious minima.
- Configuration 4:4:4:4: four squares, at different heights $\pm h_1, \pm h_2$, with 45 degrees between consecutive squares.

Figure: Squares on the sphere S^2

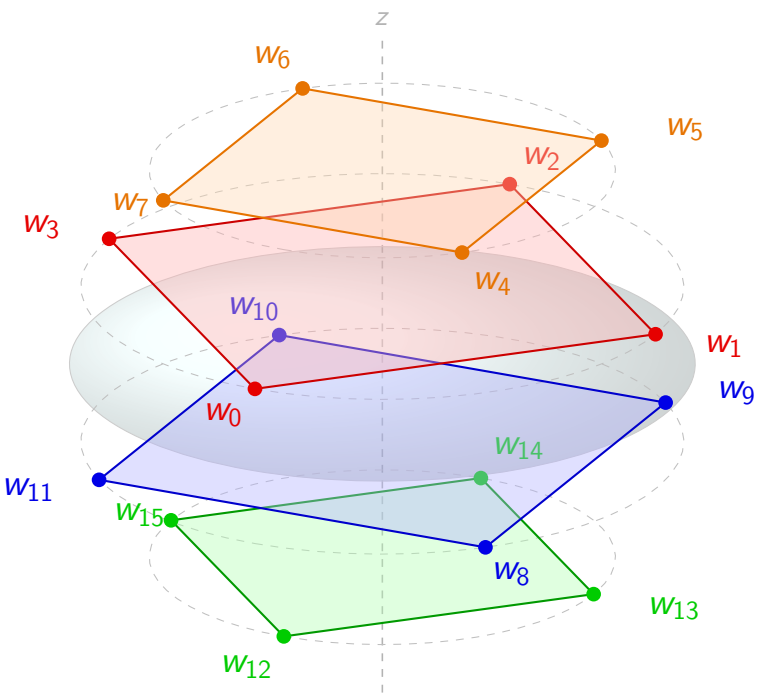
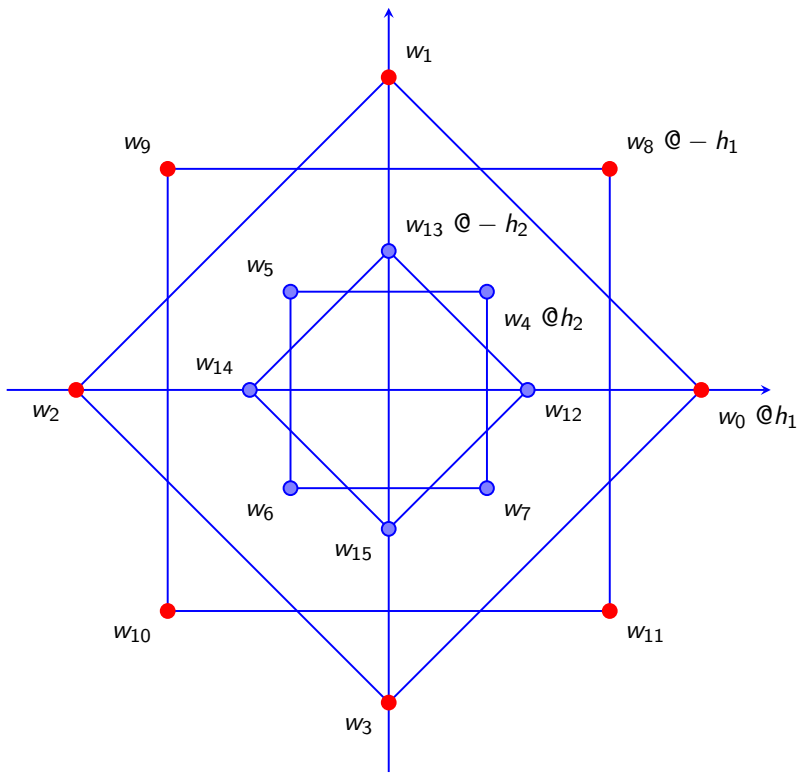


Figure: Configuration from above



Ideal of Critical Heights

We do not known the height values that make 4:4:4:4 critical. We will work modulo the polynomial system that gives these heights.

Ideal of Critical Heights

We do not known the height values that make 4:4:4:4 critical. We will work modulo the polynomial system that gives these heights. Lagrangian function of the optimization problem:

$$L(w,\lambda) := -\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \log\left(\|w_i - w_j\|_2^2\right) + \sum_{i=1}^{n} \lambda_i \left(\|w_i\|_2^2 - 1\right).$$

Ideal of Critical Heights

We do not known the height values that make 4:4:4:4 critical. We will work modulo the polynomial system that gives these heights. Lagrangian function of the optimization problem:

$$L(w,\lambda) := -\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \log\left(\|w_i - w_j\|_2^2\right) + \sum_{i=1}^{n}\lambda_i\left(\|w_i\|_2^2 - 1\right).$$

Critical configurations (null gradient condition $\frac{\partial L}{\partial w_i}(w) = \vec{0}$):

$$(n-1)w_i - \sum_{j=1,j\neq i}^{n} \frac{w_i - w_j}{1 - {w_i}^T w_j} = \vec{0}, \quad \forall\ i = 1,\ldots,n.$$

Ideal of Critical Heights

We do not known the height values that make 4:4:4:4 critical. We will work modulo the polynomial system that gives these heights. Lagrangian function of the optimization problem:

$$L(w,\lambda) := -\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \log\left(\|w_i - w_j\|_2^2\right) + \sum_{i=1}^{n}\lambda_i\left(\|w_i\|_2^2 - 1\right).$$

Critical configurations (null gradient condition $\frac{\partial L}{\partial w_i}(w) = \vec{0}$):

$$(n-1)w_i - \sum_{j=1,j\neq i}^{n} \frac{w_i - w_j}{1 - w_i{}^T w_j} = \vec{0}, \quad \forall\ i = 1,\ldots,n.$$

Critical conditions at configuration 4:4:4:4, in terms of the heights:

$$@w_0: \quad \frac{h_1\,(h_1+h_2)\,p(h_1,h_2)}{\sqrt{(1-h_1)\,(1+h_1)}\,q(h_1,h_2)} = 0, \quad \frac{(h_1+h_2)\,p(h_1,h_2)}{q(h_1,h_2)} = 0,$$

$$@w_4: \quad \frac{h_2\,(h_1+h_2)\,p(h_2,h_1)}{\sqrt{(1-h_2)\,(1+h_2)}\,q(h_2,h_1)} = 0, \quad \frac{(h_1+h_2)\,p(h_2,h_1)}{q(h_2,h_1)} = 0;$$

Ideal of Critical Heights

We do not known the height values that make 4:4:4:4 critical. We will work modulo the polynomial system that gives these heights. Lagrangian function of the optimization problem:

$$L(w,\lambda) := -\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \log\left(\|w_i - w_j\|_2^2\right) + \sum_{i=1}^{n}\lambda_i\left(\|w_i\|_2^2 - 1\right).$$

Critical configurations (null gradient condition $\frac{\partial L}{\partial w_i}(w) = \vec{0}$):

$$(n-1)w_i - \sum_{j=1,j\neq i}^{n} \frac{w_i - w_j}{1 - w_i{}^T w_j} = \vec{0}, \quad \forall\ i = 1,\ldots,n.$$

Critical conditions at configuration 4:4:4:4, in terms of the heights:

$$@w_0: \quad \frac{h_1\,(h_1+h_2)\,p(h_1,h_2)}{\sqrt{(1-h_1)\,(1+h_1)}\,q(h_1,h_2)} = 0, \quad \frac{(h_1+h_2)\,p(h_1,h_2)}{q(h_1,h_2)} = 0,$$

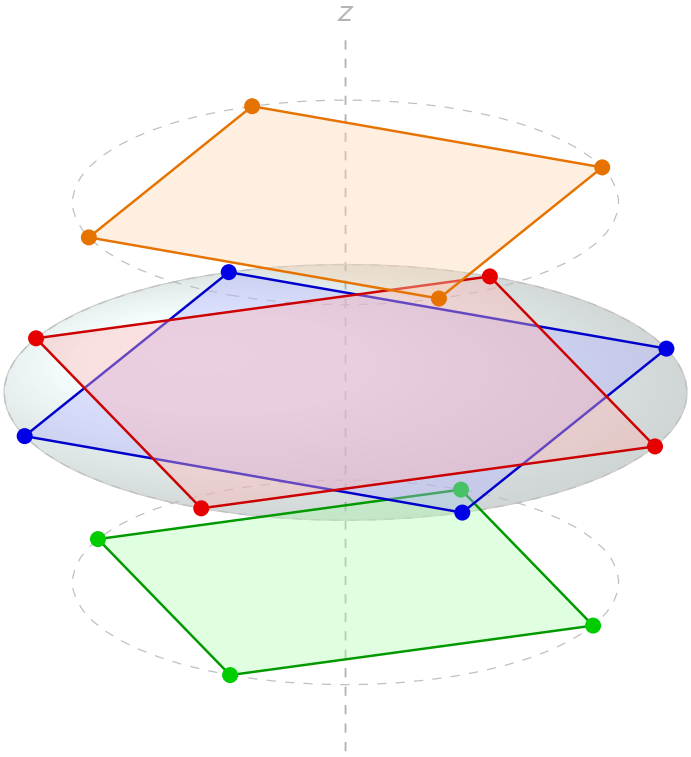
$$@w_4: \quad \frac{h_2\,(h_1+h_2)\,p(h_2,h_1)}{\sqrt{(1-h_2)\,(1+h_2)}\,q(h_2,h_1)} = 0, \quad \frac{(h_1+h_2)\,p(h_2,h_1)}{q(h_2,h_1)} = 0;$$

$$p(h_1,h_2) := 15h_1^9h_2^3 + 15h_1^9h_2 + 13h_1^8h_2^4 - 26h_1^8h_2^2 + 13h_1^8 + 44h_1^7h_2^3 + 44h_1^7h_2 + 65h_1^6h_2^4 - 130h_1^6h_2^2 +$$
$$+65h_1^6 - 134h_1^5h_2^3 - 134h_1^5h_2 - 3h_1^4h_2^4 + 6h_1^4h_2^2 - 3h_1^4 + 76h_1^3h_2^3 + 76h_1^3h_2 - 25h_1^2h_2^4 + 50h_1^2h_2^2 - 25h_1^2 - h_1h_2^3 - h_1h_2 - 2h_2^4 + 4h_2^2 - 2$$

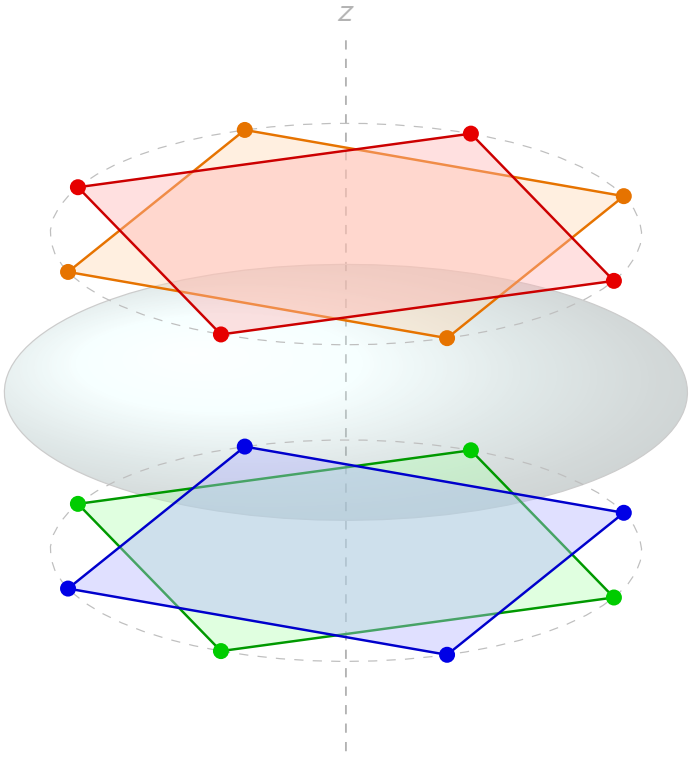
$$q(h_1,h_2) := (h_1+h_2)^2\,(h_1h_2+1)\left(h_1^4 + 6h_1^2 + 1\right)\left(h_1^2h_2^2 + h_1^2 - 4h_1h_2 + h_2^2 + 1\right).$$

Ideal of Critical Heights - Exclude "degenerate" cases

$h_1 = 0, h_2 \neq 0$

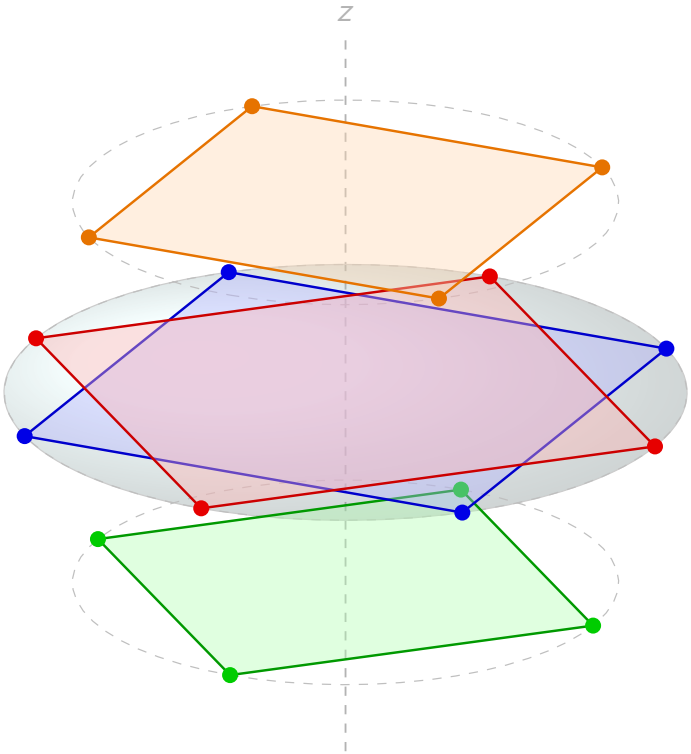


$h_1 = h_2 \neq 0$

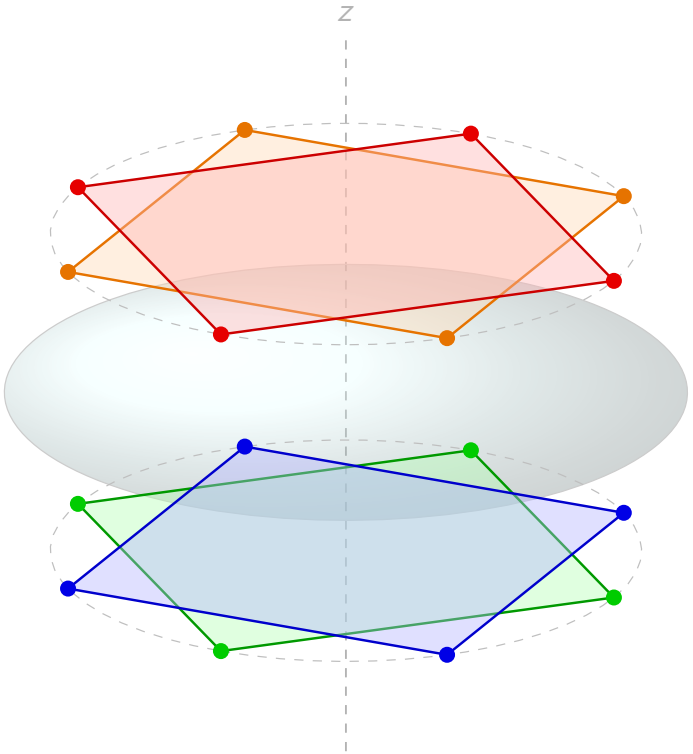


Ideal of Critical Heights - Exclude "degenerate" cases

$h_1 = 0, h_2 \neq 0$



$h_1 = h_2 \neq 0$



Ideal of critical heights (with auxiliary variables s_i):

$$p(h_1, h_2) = 0, \quad p(h_2, h_1) = 0, \quad s_1 q(h_1, h_2) = 1, \quad s_2 q(h_2, h_1) = 1,$$

$$s_3(h_1 - h_2) = 1, \quad s_4 h_1 h_2 (h_1 + h_2)\left(1 - h_1^2\right)\left(1 - h_2^2\right) - 1 = 0.$$

Real critical heights

Real critical heights

- Gröbner basis of the ideal of critical heights gives: zero dimensional ideal with 40 solutions in $\mathbb{C}$.

Real critical heights

- Gröbner basis of the ideal of critical heights gives: zero dimensional ideal with 40 solutions in $\mathbb{C}$.
- Using *Macaulay2* package *RealRoots*: exactly 4 real solutions.

Real critical heights

- Gröbner basis of the ideal of critical heights gives: zero dimensional ideal with 40 solutions in $\mathbb{C}$.
- Using *Macaulay2* package *RealRoots*: exactly 4 real solutions.
- Using interval arithmetic within *msolve*, we bound **exactly** the coordinates of each real solution, obtaining interval bounds:

$$\{h_1 \in [a,b], \quad h_2 \in [c,d]\}, \quad \{h_1 \in [c,d], \quad h_2 \in [a,b]\}.$$

$$\{h_1 \in [-d,-c], \quad h_2 \in [-b,-a]\}, \quad \{h_1 \in [-b,-a], \quad h_2 \in [-d,-c]\},$$

$$a = -438836440459385024467559303272359720231083 5/2^{142} \simeq -0.787,$$

$$b = -219418220229692512233779651636179860115541 7/2^{141} \simeq -0.787,$$

$$c = -166468600618499949666428773811293928 93/2^{126} \simeq -0.196$$

$$d = -166468600618499949666428773811293926 33/2^{126} \simeq -0.196.$$

Real critical heights

- Gröbner basis of the ideal of critical heights gives: zero dimensional ideal with 40 solutions in $\mathbb{C}$.
- Using *Macaulay2* package *RealRoots*: exactly 4 real solutions.
- Using interval arithmetic within *msolve*, we bound **exactly** the coordinates of each real solution, obtaining interval bounds:

$$\{h_1 \in [a,b], \quad h_2 \in [c,d]\}, \quad \{h_1 \in [c,d], \quad h_2 \in [a,b]\}.$$

$$\{h_1 \in [-d,-c], \quad h_2 \in [-b,-a]\}, \quad \{h_1 \in [-b,-a], \quad h_2 \in [-d,-c]\},$$

$$a = -438836440459385024467559303272359720231083 5/2^{142} \simeq -0.787,$$

$$b = -219418220229692512233779651636179860115541 7/2^{141} \simeq -0.787,$$

$$c = -166468600618499949666428773811293928 93/2^{126} \simeq -0.196$$

$$d = -166468600618499949666428773811293926 33/2^{126} \simeq -0.196.$$

- These exact bounds give approximate coordinates (h_1, h_2):

$$(-0.787, -0.196), \quad (-0.196, -0.787), \quad (0.196, 0.787), \quad (0.787, 0.196).$$

Real critical heights

- Gröbner basis of the ideal of critical heights gives: zero dimensional ideal with 40 solutions in $\mathbb{C}$.
- Using *Macaulay2* package *RealRoots*: exactly 4 real solutions.
- Using interval arithmetic within *msolve*, we bound **exactly** the coordinates of each real solution, obtaining interval bounds:

$$\{h_1 \in [a,b], \quad h_2 \in [c,d]\}, \quad \{h_1 \in [c,d], \quad h_2 \in [a,b]\}.$$

$$\{h_1 \in [-d,-c], \quad h_2 \in [-b,-a]\}, \quad \{h_1 \in [-b,-a], \quad h_2 \in [-d,-c]\},$$

$$a = -438836440459385024467559303272359720231083 5/2^{142} \simeq -0.787,$$

$$b = -219418220229692512233779651636179860115541 7/2^{141} \simeq -0.787,$$

$$c = -166468600618499949666428773811293928 93/2^{126} \simeq -0.196$$

$$d = -166468600618499949666428773811293926 33/2^{126} \simeq -0.196.$$

- These exact bounds give approximate coordinates (h_1, h_2):

$$(-0.787, -0.196), \quad (-0.196, -0.787), \quad (0.196, 0.787), \quad (0.787, 0.196).$$

- By symmetry: system has only one "kind" of real solution.

Real critical heights

- Gröbner basis of the ideal of critical heights gives: zero dimensional ideal with 40 solutions in $\mathbb{C}$.
- Using *Macaulay2* package *RealRoots*: exactly 4 real solutions.
- Using interval arithmetic within *msolve*, we bound **exactly** the coordinates of each real solution, obtaining interval bounds:

$$\{h_1 \in [a,b], \quad h_2 \in [c,d]\}, \quad \{h_1 \in [c,d], \quad h_2 \in [a,b]\}.$$

$$\{h_1 \in [-d,-c], \quad h_2 \in [-b,-a]\}, \quad \{h_1 \in [-b,-a], \quad h_2 \in [-d,-c]\},$$

$$a = -438836440459385024467559303272359720231083 5/2^{142} \simeq -0.787,$$

$$b = -219418220229692512233779651636179860115541 7/2^{141} \simeq -0.787,$$

$$c = -166468600618499949666428773811293928 93/2^{126} \simeq -0.196$$

$$d = -166468600618499949666428773811293926 33/2^{126} \simeq -0.196.$$

- These exact bounds give approximate coordinates (h_1, h_2):

$$(-0.787, -0.196), \quad (-0.196, -0.787), \quad (0.196, 0.787), \quad (0.787, 0.196).$$

- By symmetry: system has only one "kind" of real solution.
- Work modulo the ideal of critical heights, to restrict results (about real critical configurations) to configuration 4:4:4:4.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.
- 48×48 matrix, which depends on variables h_1 and h_2.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.
- 48×48 matrix, which depends on variables h_1 and h_2.
- **Positive definite** in **feasible** directions $\Rightarrow$ **local minimum**.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.
- 48×48 matrix, which depends on variables h_1 and h_2.
- **Positive definite** in **feasible** directions $\Rightarrow$ **local minimum**.
- Feasible directions are the ones orthogonal to the points w_i.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.
- 48×48 matrix, which depends on variables h_1 and h_2.
- **Positive definite** in **feasible** directions $\Rightarrow$ **local minimum**.
- Feasible directions are the ones orthogonal to the points w_i.
- Rotational symmetry $\Rightarrow$ Hessian has at least three zero eigenvalues, corresponding to rotations of the sphere.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.
- 48×48 matrix, which depends on variables h_1 and h_2.
- **Positive definite** in **feasible** directions $\Rightarrow$ **local minimum**.
- Feasible directions are the ones orthogonal to the points w_i.
- Rotational symmetry $\Rightarrow$ Hessian has at least three zero eigenvalues, corresponding to rotations of the sphere.
- Sufficient to prove Hessian positive definite in feasible directions that do not correspond to rotations.

Hessian of the Lagrangian at 4:4:4:4

- We will use the Hessian to calssify 4:4:4:4 as a local minimum.
- 48×48 matrix, which depends on variables h_1 and h_2.
- **Positive definite** in **feasible** directions $\Rightarrow$ **local minimum**.
- Feasible directions are the ones orthogonal to the points w_i.
- Rotational symmetry $\Rightarrow$ Hessian has at least three zero eigenvalues, corresponding to rotations of the sphere.
- Sufficient to prove Hessian positive definite in feasible directions that do not correspond to rotations.
- We will see that Hessian is positive definite in **all** directions of $\mathbb{R}^{48}$ (except the ones corresponding to rotations of the sphere)

Numerical eigenvalues

- Using floating point arithmetic: evaluate the Hessian in the approximate heights and obtain approximate eigenvalues

Numerical eigenvalues

- Using floating point arithmetic: evaluate the Hessian in the approximate heights and obtain approximate eigenvalues

Value	0	0.175	0.519	0.905	1.55	2.70	3.27	4.43	4.44
Alg. mult.	3	2	2	2	1	1	2	1	2
Value	5.48	6.15	6.28	7.18	8.81	9.05	9.19	10.6	10.6
Alg. mult.	1	2	2	2	2	1	2	2	1
Value	11.7	12.3	13.5	14.1	14.5	14.8	15	30	
Alg. mult.	2	1	1	2	2	2	6	1	

Numerical eigenvalues

- Using floating point arithmetic: evaluate the Hessian in the approximate heights and obtain approximate eigenvalues

Value	0	0.175	0.519	0.905	1.55	2.70	3.27	4.43	4.44
Alg. mult.	3	2	2	2	1	1	2	1	2
Value	5.48	6.15	6.28	7.18	8.81	9.05	9.19	10.6	10.6
Alg. mult.	1	2	2	2	2	1	2	2	1
Value	11.7	12.3	13.5	14.1	14.5	14.8	15	30	
Alg. mult.	2	1	1	2	2	2	6	1	

- All **positive** (even in non feasible directions), except three which are zero and correspond to the directions of rotation.

Numerical eigenvalues

- Using floating point arithmetic: evaluate the Hessian in the approximate heights and obtain approximate eigenvalues

Value	0	0.175	0.519	0.905	1.55	2.70	3.27	4.43	4.44
Alg. mult.	3	2	2	2	1	1	2	1	2
Value	5.48	6.15	6.28	7.18	8.81	9.05	9.19	10.6	10.6
Alg. mult.	1	2	2	2	2	1	2	2	1
Value	11.7	12.3	13.5	14.1	14.5	14.8	15	30	
Alg. mult.	2	1	1	2	2	2	6	1	

- All **positive** (even in non feasible directions), except three which are zero and correspond to the directions of rotation.
- **Objective:** prove this result in an exact way.

Numerical eigenvalues

- Using floating point arithmetic: evaluate the Hessian in the approximate heights and obtain approximate eigenvalues

Value	0	0.175	0.519	0.905	1.55	2.70	3.27	4.43	4.44
Alg. mult.	3	2	2	2	1	1	2	1	2
Value	5.48	6.15	6.28	7.18	8.81	9.05	9.19	10.6	10.6
Alg. mult.	1	2	2	2	2	1	2	2	1
Value	11.7	12.3	13.5	14.1	14.5	14.8	15	30	
Alg. mult.	2	1	1	2	2	2	6	1	

- All **positive** (even in non feasible directions), except three which are zero and correspond to the directions of rotation.
- **Objective:** prove this result in an exact way.
- But: Hessian is dense symbolic matrix, with $48^2 = 2304$ entries which are rational functions.

Numerical eigenvalues

- Using floating point arithmetic: evaluate the Hessian in the approximate heights and obtain approximate eigenvalues

Value	0	0.175	0.519	0.905	1.55	2.70	3.27	4.43	4.44
Alg. mult.	3	2	2	2	1	1	2	1	2
Value	5.48	6.15	6.28	7.18	8.81	9.05	9.19	10.6	10.6
Alg. mult.	1	2	2	2	2	1	2	2	1
Value	11.7	12.3	13.5	14.1	14.5	14.8	15	30	
Alg. mult.	2	1	1	2	2	2	6	1	

- All **positive** (even in non feasible directions), except three which are zero and correspond to the directions of rotation.
- **Objective:** prove this result in an exact way.
- But: Hessian is dense symbolic matrix, with $48^2 = 2304$ entries which are rational functions.
- **How?** Exploit symmetries of configuration 4:4:4:4 to block diagonalize the Hessian and classify each block separately.

Group of Symmetries of the Square Antiprism

- Transformations that leave the shape unchanged.

Group of Symmetries of the Square Antiprism

- Transformations that leave the shape unchanged.
- Conf. 4:4:4:4 has, at least, symmetries of square antiprism

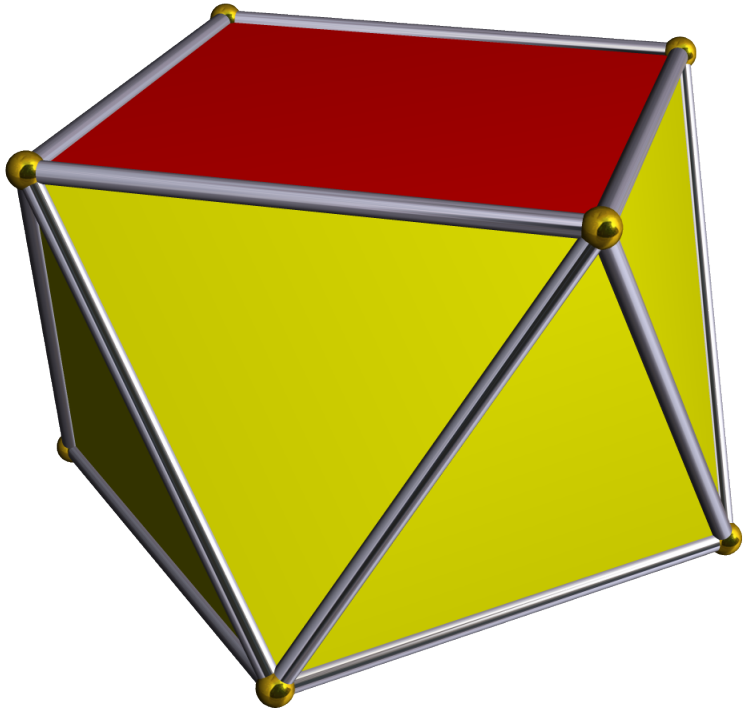


Figure: Square Antiprism (from Wikipedia).

Group of Symmetries of the Square Antiprism

- Transformations that leave the shape unchanged.
- Conf. 4:4:4:4 has, at least, symmetries of square antiprism

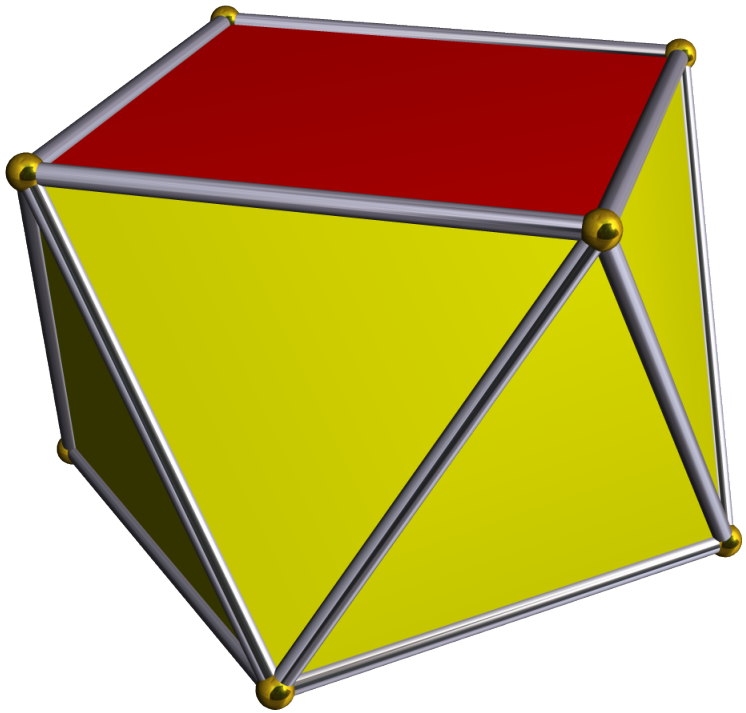


Figure: Square Antiprism (from Wikipedia).

- Group of symmetries of square antiprism is D_{4d}, isomorphic to D_8 (symmetries of regular polygon of 8 vertices in the plane)

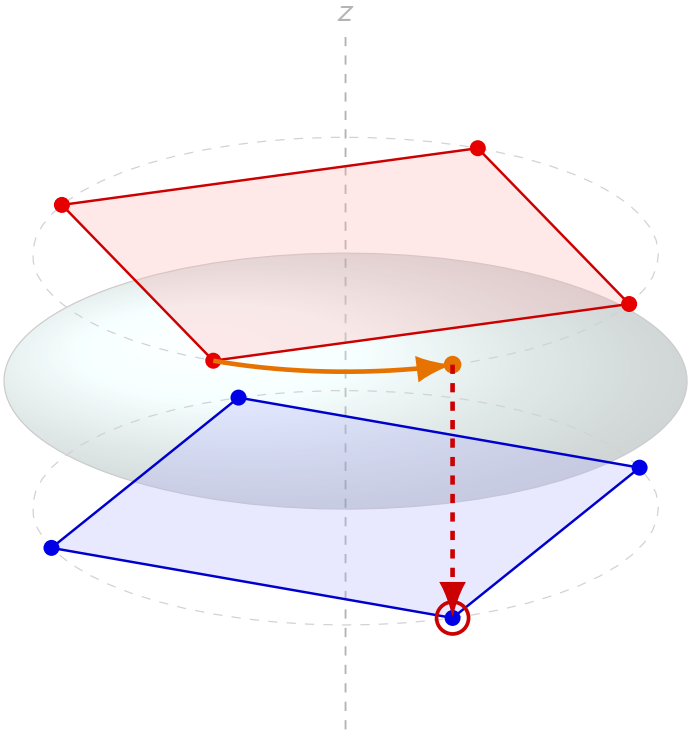
Group of Symmetries of the Square Antiprism

To generate D_{4d}, define the following transformations in $\mathbb{R}^3$:

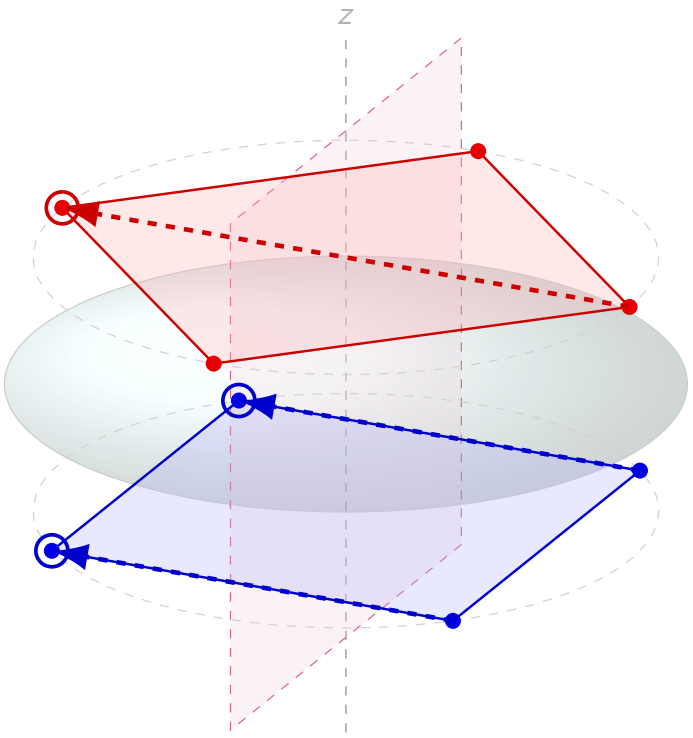
Group of Symmetries of the Square Antiprism

To generate D_{4d}, define the following transformations in $\mathbb{R}^3$:

Composition: $S_z \circ R_{45}$



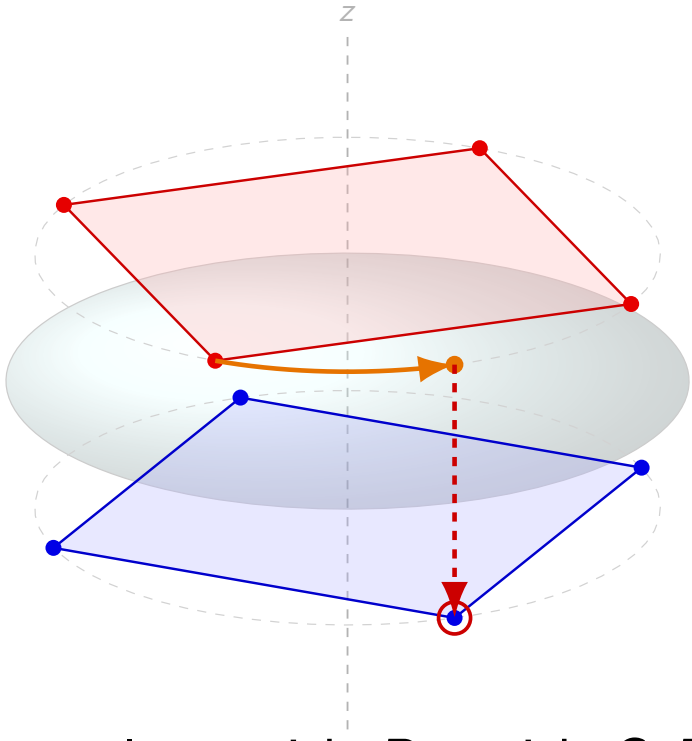
Reflection: S_y



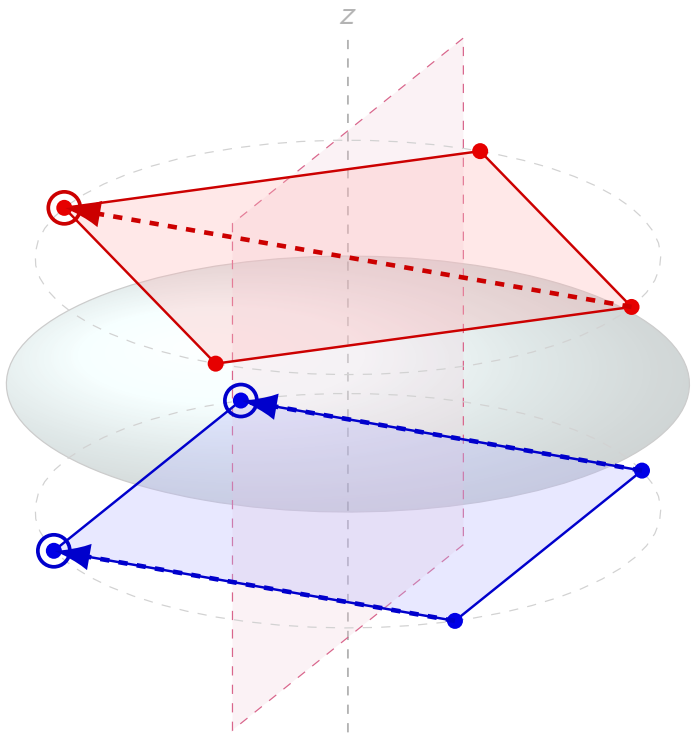
Group of Symmetries of the Square Antiprism

To generate D_{4d}, define the following transformations in $\mathbb{R}^3$:

Composition: $S_z \circ R_{45}$



Reflection: S_y



In analogy with D_8, with $S_z R_{45}$ playing the role of the rotation in the plane, and S_y of a reflection, we can describe D_{4d} as:

$$D_{4d} = (S_z R_{45}, S_y), \quad (S_z R_{45})^8 = I_3, \quad S_y^2 = I_3, \quad (S_y (S_z R_{45}))^2 = I_3.$$

Linear Representations of finite groups G

Linear Representations of finite groups G

- Given a finite group G and a vector space V of dimension N, a linear representation of G is a pair (ρ, V), such that:
  1. $\rho : G \to GL(V)$ associates to each $g \in G$ an invertible linear transformation ($N \times N$ matrix) $\rho(g) : V \to V$.
  2. ρ is a group homomorphism:

$$\rho(g_1 * g_2) = \rho(g_1)\rho(g_2), \quad \forall\ g_1, g_2 \in G.$$

Linear Representations of finite groups G

- Given a finite group G and a vector space V of dimension N, a linear representation of G is a pair (ρ, V), such that:
  1. $\rho : G \to GL(V)$ associates to each $g \in G$ an invertible linear transformation ($N \times N$ matrix) $\rho(g) : V \to V$.
  2. ρ is a group homomorphism:

$$\rho(g_1 * g_2) = \rho(g_1)\rho(g_2), \quad \forall\ g_1, g_2 \in G.$$

- Every group G has "irreducible representations" (ρ_i, V_i), which are "building blocks" of any representation (ρ, V) of G:

$\rho = \rho_1^{m_1} \oplus \ldots \oplus \rho_k^{m_k}$, $m_i \geq 0$, given by dot product of "characters".

Linear Representations of finite groups G

- Given a finite group G and a vector space V of dimension N, a linear representation of G is a pair (ρ, V), such that:
  1. $\rho : G \to GL(V)$ associates to each $g \in G$ an invertible linear transformation ($N \times N$ matrix) $\rho(g) : V \to V$.
  2. ρ is a group homomorphism:

$$\rho(g_1 * g_2) = \rho(g_1)\rho(g_2), \quad \forall\ g_1, g_2 \in G.$$

- Every group G has "irreducible representations" (ρ_i, V_i), which are "building blocks" of any representation (ρ, V) of G:

  $\rho = \rho_1^{m_1} \oplus \ldots \oplus \rho_k^{m_k}$, $m_i \geq 0$, given by dot product of "characters".

- By Schur's Lemma: consider a linear transformation (matrix) $H : V \to V$ that commutes with a representation (ρ, V):

$$H\rho(g) = \rho(g)H, \ \forall\ g \in G.$$

  H is block diagonalized in a basis of the irreducible representations that "build" ρ.

Block Diagonal Hessian

We define a representation $(\rho, \mathbb{R}^{48})$ of D_{4d}, such that:

Block Diagonal Hessian

We define a representation $(\rho, \mathbb{R}^{48})$ of D_{4d}, such that:

Theorem

The Hessian of the Lagrangian, evaluated at configuration 4:4:4:4, commutes with our representation $(\rho, \mathbb{R}^{48})$.

Block Diagonal Hessian

We define a representation $(\rho, \mathbb{R}^{48})$ of D_{4d}, such that:

Theorem

The Hessian of the Lagrangian, evaluated at configuration 4:4:4:4, commutes with our representation $(\rho, \mathbb{R}^{48})$.

Corollary

A basis where the 48×48 *Hessian is* **block-diagonal**: *2 blocks of size* 4×4, *2 blocks of size* 2×2, *6 blocks of size* 6×6.

$$H_L \simeq \begin{bmatrix}
* & * & * & * & & & & & & & & & & & & & & & & & & & & \\
* & * & * & * & & & & & & & & & & & & & & & & & & & & \\
* & * & * & * & & & & & & & & & & & & & & & & & & & & \\
* & * & * & * & & & & & & & & & & & & & & & & & & & & \\
 & & & & * & * & * & * & & & & & & & & & & & & & & & & & \\
 & & & & * & * & * & * & & & & & & & & & & & & & & & & & \\
 & & & & * & * & * & * & & & & & & & & & & & & & & & & & \\
 & & & & * & * & * & * & & & & & & & & & & & & & & & & & \\
 & & & & & & & & * & * & & & & & & & & & & & & & & & \\
 & & & & & & & & * & * & & & & & & & & & & & & & & & \\
 & & & & & & & & & & * & * & & & & & & & & & & & & & \\
 & & & & & & & & & & * & * & & & & & & & & & & & & & \\
 & & & & & & & & & & & & * & * & & & & & & & & & & & \\
 & & & & & & & & & & & & * & * & & & & & & & & & & & \\
 & & & & & & & & & & & & & & * & * & * & * & * & * & & & & & \\
 & & & & & & & & & & & & & & * & * & * & * & * & * & & & & & \\
 & & & & & & & & & & & & & & * & * & * & * & * & * & & & & & \\
 & & & & & & & & & & & & & & * & * & * & * & * & * & & & & & \\
 & & & & & & & & & & & & & & * & * & * & * & * & * & & & & & \\
 & & & & & & & & & & & & & & * & * & * & * & * & * & & & & &
\end{bmatrix}$$

Block Diagonal Hessian

To obtain this basis of $\mathbb{R}^{48}$, were H_L is block-diagonal, we need to:

Block Diagonal Hessian

To obtain this basis of $\mathbb{R}^{48}$, were H_L is block-diagonal, we need to:

- Define the representation $(\rho, \mathbb{R}^{48})$ of the theorem.

Block Diagonal Hessian

To obtain this basis of $\mathbb{R}^{48}$, were H_L is block-diagonal, we need to:

- Define the representation $(\rho, \mathbb{R}^{48})$ of the theorem.
- Find the irreps. of D_{4d} that "build" $(\rho, \mathbb{R}^{48})$: dot product of **character functions** of irreps. of D_{4d} and our representation.

Block Diagonal Hessian

To obtain this basis of $\mathbb{R}^{48}$, were H_L is block-diagonal, we need to:

- Define the representation $(\rho, \mathbb{R}^{48})$ of the theorem.
- Find the irreps. of D_{4d} that "build" $(\rho, \mathbb{R}^{48})$: dot product of **character functions** of irreps. of D_{4d} and our representation.
- Project a basis of $\mathbb{R}^{48}$ on each of these irreps.

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

- Let $w = (w_0, w_1, \ldots, w_{15}) \in \left(\mathbb{R}^3\right)^{16}$, with $w_i \in \mathbb{R}^3$ the coordinate vectors of each point.

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

- Let $w = (w_0, w_1, \ldots, w_{15}) \in \left(\mathbb{R}^3\right)^{16}$, with $w_i \in \mathbb{R}^3$ the coordinate vectors of each point.
- Given $g \in D_{4d}$, we associate to g the linear transformation $\rho(g) : \mathbb{R}^{48} \to \mathbb{R}^{48}$, obtained in the following way:

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

- Let $w = (w_0, w_1, \ldots, w_{15}) \in \left(\mathbb{R}^3\right)^{16}$, with $w_i \in \mathbb{R}^3$ the coordinate vectors of each point.
- Given $g \in D_{4d}$, we associate to g the linear transformation $\rho(g) : \mathbb{R}^{48} \to \mathbb{R}^{48}$, obtained in the following way:
  1. apply g to each vector $w_i \in \mathbb{R}^3$, and

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

- Let $w = (w_0, w_1, \ldots, w_{15}) \in \left(\mathbb{R}^3\right)^{16}$, with $w_i \in \mathbb{R}^3$ the coordinate vectors of each point.
- Given $g \in D_{4d}$, we associate to g the linear transformation $\rho(g) : \mathbb{R}^{48} \to \mathbb{R}^{48}$, obtained in the following way:
  1. apply g to each vector $w_i \in \mathbb{R}^3$, and
  2. permute the resulting vectors in such a way that: $\rho(g)(w) = w$.

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

- Let $w = (w_0, w_1, \ldots, w_{15}) \in \left(\mathbb{R}^3\right)^{16}$, with $w_i \in \mathbb{R}^3$ the coordinate vectors of each point.
- Given $g \in D_{4d}$, we associate to g the linear transformation $\rho(g) : \mathbb{R}^{48} \to \mathbb{R}^{48}$, obtained in the following way:
  1. apply g to each vector $w_i \in \mathbb{R}^3$, and
  2. permute the resulting vectors in such a way that: $\rho(g)(w) = w$.

$$\rho(S_y) = \left(\begin{array}{c|c|c|c} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & A \end{array}\right), \quad A := \left(\begin{array}{c|c|c|c} S_y & 0 & 0 & 0 \\ 0 & 0 & 0 & S_y \\ 0 & 0 & S_y & 0 \\ 0 & S_y & 0 & 0 \end{array}\right), \quad B := \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & S_y \\ 0 & 0 & S_y & 0 \\ 0 & S_y & 0 & 0 \\ S_y & 0 & 0 & 0 \end{array}\right).$$

$$\rho(S_z R_{45}) = \left(\begin{array}{c|c|c|c} 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \\ D & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{array}\right), \quad C := \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & M \\ M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \end{array}\right), \quad D := \left(\begin{array}{c|c|c|c} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{array}\right).$$

Our Representation of D_{4d} in $\left(\mathbb{R}^3\right)^{16}$

- Let $w = (w_0, w_1, \ldots, w_{15}) \in \left(\mathbb{R}^3\right)^{16}$, with $w_i \in \mathbb{R}^3$ the coordinate vectors of each point.
- Given $g \in D_{4d}$, we associate to g the linear transformation $\rho(g) : \mathbb{R}^{48} \to \mathbb{R}^{48}$, obtained in the following way:
  1. apply g to each vector $w_i \in \mathbb{R}^3$, and
  2. permute the resulting vectors in such a way that: $\rho(g)(w) = w$.

$$\rho(S_y) = \left(\begin{array}{c|c|c|c} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & A \end{array}\right), \quad A := \left(\begin{array}{c|c|c|c} S_y & 0 & 0 & 0 \\ 0 & 0 & 0 & S_y \\ 0 & 0 & S_y & 0 \\ 0 & S_y & 0 & 0 \end{array}\right), \quad B := \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & S_y \\ 0 & 0 & S_y & 0 \\ 0 & S_y & 0 & 0 \\ S_y & 0 & 0 & 0 \end{array}\right).$$

$$\rho(S_z R_{45}) = \left(\begin{array}{c|c|c|c} 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \\ D & 0 & 0 & 0 \\ 0 & C & 0 & 0 \end{array}\right), \quad C := \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & M \\ M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \end{array}\right), \quad D := \left(\begin{array}{c|c|c|c} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{array}\right).$$

$$\rho\left(\left(S_z R_{45}\right)^k\right) = \rho\left(S_z R_{45}\right)^k, \; \rho\left(\left(S_z R_{45}\right)^k S_y\right) = \rho\left(S_z R_{45}\right)^k \rho(S_y), \quad \forall\ k.$$

Characters of our Representation

Characters of our Representation

- The **character function** of our representation is, by definition, the trace of each of the previous 48×48 matrices:

$$\chi : D_{4d} \to \mathbb{R} \quad / \quad \chi(g) := \text{trace}\left(\rho(g)\right), \ \forall \ g \in D_{4d}.$$

Characters of our Representation

- The **character function** of our representation is, by definition, the trace of each of the previous 48×48 matrices:

$$\chi : D_{4d} \to \mathbb{R} \quad / \quad \chi(g) := \text{trace}\left(\rho(g)\right), \ \forall \ g \in D_{4d}.$$

- Traces are calculated exactly using *SymPy*, and given by:

$$\chi(e) = 48, \quad \chi\left(\rho(r^k s)\right) = \begin{cases} 4, \ \forall \ k = 0, 2, 4, 6 \\ 0, \ \forall \ k = 1, 3, 5, 7 \end{cases},$$

$$\chi\left(\rho(r^k)\right) = 0, \ \forall \ k \in \{1, \ldots, 7\}.$$

Number of Copies of each Irreducible Representation

Number of Copies of each Irreducible Representation

- D_{4d} has 7 irreps. ρ_i: 4 of dimension 1 and 3 of dimension 2.

Number of Copies of each Irreducible Representation

- D_{4d} has 7 irreps. ρ_i: 4 of dimension 1 and 3 of dimension 2.
- Our representation of D_{4d} may be written in terms of these irreps.:

$$\rho = \rho_1^{m_1} \oplus \rho_2^{m_2} \oplus \ldots \oplus \rho_7^{m_7}.$$

Number of Copies of each Irreducible Representation

- D_{4d} has 7 irreps. ρ_i: 4 of dimension 1 and 3 of dimension 2.
- Our representation of D_{4d} may be written in terms of these irreps.:

$$\rho = \rho_1^{m_1} \oplus \rho_2^{m_2} \oplus \ldots \oplus \rho_7^{m_7}.$$

- Number of copies of irrep. ρ_i in our representation is given by:

$$m_i = \langle \chi, \chi_i \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi_i(g).$$

Number of Copies of each Irreducible Representation

- D_{4d} has 7 irreps. ρ_i: 4 of dimension 1 and 3 of dimension 2.
- Our representation of D_{4d} may be written in terms of these irreps.:

$$\rho = \rho_1^{m_1} \oplus \rho_2^{m_2} \oplus \ldots \oplus \rho_7^{m_7}.$$

- Number of copies of irrep. ρ_i in our representation is given by:

$$m_i = \langle \chi, \chi_i \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi_i(g).$$

irrep.	irrep. copies $\langle \chi, \chi_i \rangle$	irrep. dimension	copies $\times$ dimension
ρ_1	$3 + 1 = 4$	1	4
ρ_2	$3 + 1 = 4$	1	4
ρ_3	$3 - 1 = 2$	1	2
ρ_4	$3 - 1 = 2$	1	2
ρ_5	$3 \times 2 + 0 = 6$	2	12
ρ_6	$3 \times 2 + 0 = 6$	2	12
ρ_7	$3 \times 2 + 0 = 6$	2	12
		Total dimension	48

Diagonal Blocks Classification

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

1. Add the characteristic polynomial $p(t) := \det(A - tI)$ to the ideal of critical heights I, obtaining the ideal $I + (p)$.

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

1. Add the characteristic polynomial $p(t) := \det(A - tI)$ to the ideal of critical heights I, obtaining the ideal $I + (p)$.
2. $p(t)$ is of degree at most the size of the block in terms of the entries A_{ij} of A (which we introduce as new variables).

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

1. Add the characteristic polynomial $p(t) := \det(A - tI)$ to the ideal of critical heights I, obtaining the ideal $I + (p)$.
2. $p(t)$ is of degree at most the size of the block in terms of the entries A_{ij} of A (which we introduce as new variables).
3. Project $I + (p)$ to $I \cap \mathbb{Q}[h_1, h_2, t]$, calculating a **Gröbner basis** with *msolve*, using an elimination monomial order (two blocks of variables, GRevLex inside each block).

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

1. Add the characteristic polynomial $p(t) := \det(A - tI)$ to the ideal of critical heights I, obtaining the ideal $I + (p)$.
2. $p(t)$ is of degree at most the size of the block in terms of the entries A_{ij} of A (which we introduce as new variables).
3. Project $I + (p)$ to $I \cap \mathbb{Q}[h_1, h_2, t]$, calculating a **Gröbner basis** with *msolve*, using an elimination monomial order (two blocks of variables, GRevLex inside each block).
4. Find **exact** bounds for the coordinates of each **real** solution of $I \cap \mathbb{Q}[h_1, h_2, t]$, using **interval arithmetic** via *msolve*.

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

1. Add the characteristic polynomial $p(t) := \det(A - tI)$ to the ideal of critical heights I, obtaining the ideal $I + (p)$.
2. $p(t)$ is of degree at most the size of the block in terms of the entries A_{ij} of A (which we introduce as new variables).
3. Project $I + (p)$ to $I \cap \mathbb{Q}[h_1, h_2, t]$, calculating a **Gröbner basis** with *msolve*, using an elimination monomial order (two blocks of variables, GRevLex inside each block).
4. Find **exact** bounds for the coordinates of each **real** solution of $I \cap \mathbb{Q}[h_1, h_2, t]$, using **interval arithmetic** via *msolve*.
5. By definition, the values of t are the eigenvalues of the block.

Diagonal Blocks Classification

To determine the sign of the eigenvalues of each block A, we apply the following procedure to each block:

1. Add the characteristic polynomial $p(t) := \det(A - tI)$ to the ideal of critical heights I, obtaining the ideal $I + (p)$.
2. $p(t)$ is of degree at most the size of the block in terms of the entries A_{ij} of A (which we introduce as new variables).
3. Project $I + (p)$ to $I \cap \mathbb{Q}[h_1, h_2, t]$, calculating a **Gröbner basis** with *msolve*, using an elimination monomial order (two blocks of variables, GRevLex inside each block).
4. Find **exact** bounds for the coordinates of each **real** solution of $I \cap \mathbb{Q}[h_1, h_2, t]$, using **interval arithmetic** via *msolve*.
5. By definition, the values of t are the eigenvalues of the block.
6. For each t: verify that the lower bound of the interval is positive (except three of them).

A detailed example for Block 4, of size 2×2

```
R = QQ[s1, s2, s3, u0, u1, A00, A01, A11, h0, h1, t];

I = initialize critical heights ideal.

-- define sqrt{1-h0^2}, sqrt{1-h1^2} for some Aij
I = I + ideal( u0^2 - ( 1-h0^2 ), u1^2 - ( 1-h1^2 ) );

-- add characteristic polynomial
A = matrix{ {A00, A01}, {A01, A11} };
I = I + ideal( det(A - t * matrix{{1,0},{0,1}} ) );

-- Aij are rational functions in the other variables
I = I + ideal(
    A00 * (h0^18*h1^6 + ... ) - (... - 24*h1^2 + 2),
    A01 * (h0^8*h1^6 + ... ) - (... + 2*u0*u1),
    A11 * (h0^8*h1^16 + ... ) - (... - 2*h1^2 + 2)
);
```

A detailed example for Block 4, of size 2×2

- Projected ideal has variables h_0, h_1, t only.

A detailed example for Block 4, of size 2×2

- Projected ideal has variables h_0, h_1, t only.
- Reduced Gröbner basis of this ideal has 16 polynomials, with "big" coefficients:

$$37129249108850546172945902033303729582080 0t^6 - 16214149936235981704594519536121420079544956000362 5h_0^4 t - \ldots + 11821098231394534498129261022775065161027422742641387 9,$$

$$94717472216455474930984443962509514240 0h_0 h_1 t^4 + 12910787399795569535647919624114190585846712 5h_0^4 t + \ldots + 44510440502623609989810740209174328469444340 53t - 24236909582057465464703668056872429827406416527,$$

$$\vdots$$

$$58273444820670848834492382515997064425000 h_0^2 h_1^5 - 19872276148503933316980431005986760058625 h_0^5 t - \ldots + 83175354438620611368875441130479875410131 01h_1$$

A detailed example for Block 4, of size 2×2

A detailed example for Block 4, of size 2×2

- Projected ideal has exactly 8 real solutions (h_0, h_1, t); four solutions for each eigenvalue t of the 2×2 matrix.

A detailed example for Block 4, of size 2×2

- Projected ideal has exactly 8 real solutions (h_0, h_1, t); four solutions for each eigenvalue t of the 2×2 matrix.
- Remember that ideal of critical heights has exactly 4 real solutions (h_0, h_1), all of the "kind" 4:4:4:4.

A detailed example for Block 4, of size 2×2

- Projected ideal has exactly 8 real solutions (h_0, h_1, t); four solutions for each eigenvalue t of the 2×2 matrix.
- Remember that ideal of critical heights has exactly 4 real solutions (h_0, h_1), all of the "kind" 4:4:4:4.
- Coordinates of one of the eight real solutions are bounded as:

$$h_0 \in [665874402473999798665715095245175710 51/2^{128},$$
$$166468600618499949666428773811293927 63/2^{126}] \simeq [.196, .196],$$

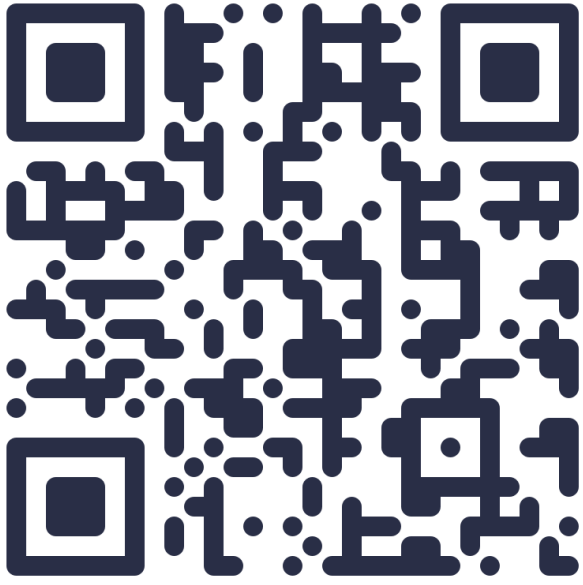
$$h_1 \in [267844507116323867472875551313696118305/2^{128},$$
$$133922253558161933736437775656848059153/2^{127}] \simeq [.787, .787],$$

$$t \in [899504581213468778226187660896150819527/2^{126},$$
$$359801832485387511290475064358460327810 9/2^{128}] \simeq [10.57, 10.57].$$

Merci de votre attention

Code and Presentation

`http://www.github.com/matiasvd`



Acknowledgments

- ANII and CAP for PhD grants.
- *msolve*, *Macaulay2* and *SymPy* developers for their great work

Hessian of the Lagrangian

1. For n points, the Hessian of the Lagrangian is a $3n \times 3n$ matrix H_L, made by blocks of 3×3 matrices, each given by:

$$(H_L)_{ii} := \frac{\partial L}{\partial w_i^2} = \sum_{j=1, j\neq i} J_f(w_i - w_j) + (n-1)\mathrm{Id}_3, \quad \forall\ i.$$

$$(H_L)_{ij} := \frac{\partial L}{\partial w_i w_j} = -J_f(w_i - w_j), \quad \forall\ i \neq j;$$

where we define an auxiliary function $f : \mathbb{R}^3 \to \mathbb{R}^3$, with Jacobian:

$$J_f(w) := \frac{-2}{\|w\|_2^4} \begin{pmatrix} -x^2+y^2+z^2 & -2xy & -2xz \\ -2xy & x^2-y^2+z^2 & -2yz \\ -2xz & -2yz & x^2+y^2-z^2 \end{pmatrix}.$$

2. Entries of H_L are rational functions (quotients of polynomials)

Characters of irreps. of D_{4d} and of our representation

Denote $r := S_z R_{45}$, $s := S_y$. Equivalence classes of D_{4d} are:

$$\{e\}, \{r, r^7\}, \{r^2, r^6\}, \{r^3, r^5\}, \{r^4\}, \{s, r^2 s, r^4 s, r^6 s\}, \{rs, r^3 s, r^5 s, r^7 s\}.$$

Table: Characters of the irreducible representations of D_{4d}.

irrep. / class	$[e]$	$[r]$	$[r^2]$	$[r^3]$	$[r^4]$	$[s]$	$[rs]$
ρ_1	1	1	1	1	1	1	1
ρ_2	1	-1	1	-1	1	1	-1
ρ_3	1	1	1	1	1	-1	-1
ρ_4	1	-1	1	-1	1	-1	1
ρ_5	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0
ρ_6	2	0	-2	0	2	0	0
ρ_7	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0

Table: Character χ of our representation ρ. $r := S_z R_{45}$, $s := S_y$

class	$[e]$	$[r]$	$[r^2]$	$[r^3]$	$[r^4]$	$[s]$	$[rs]$
χ	48	0	0	0	0	4	0

References I



Laurent Bétermin and Etienne Sandier.

Renormalized energy and asymptotic expansion of optimal logarithmic energy on the sphere.
Constructive Approximation, 47(1):39–74, 2018.



Kevin Constantineau, Carlos García-Azpeitia, Luis C García-Naranjo, and J-P Lessard.

Determination of stable branches of relative equilibria of the n-vortex problem on the sphere.
Communications in Mathematical Physics, 406(2):47, 2025.



Peter D. Dragnev.

Log-optimal configurations on the sphere.
In *Modern Trends in Constructive Function Theory*, volume 661 of *Contemporary Mathematics*, pages 41–55. American Mathematical Society, Providence, RI, 2016.



Peter D Dragnev and Oleg R Musin.

Log-optimal $(d+2)$-configurations in d-dimensions.
Transactions of the American Mathematical Society, Series B, 10, 2023.



Evguenii A Rakhmanov, Edward B Saff, and Yan M Zhou.

Electrons on the sphere.
In *Computational Methods and Function Theory 1994*, pages 293–309. World Scientific, 1995.



Michael Shub and Steve Smale.

Complexity of bezout's theorem: III. condition number and packing.
Journal of complexity, 9(1):4–14, 1993.



Steve Smale.

Mathematical problems for the next century.
The mathematical intelligencer, 20(2):7–15, 1998.